

A NOTE ON PRODUCT SETS OF RATIONALS

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ABSTRACT. Bourgain, Konyagin and Shparlinski obtained a lower bound for the size of the product set AB when A and B are sets of positive rational numbers with numerator and denominator less or equal than Q . We extend and slightly improve that lower bound using a different approach.

1. INTRODUCTION

Bourgain, Konyagin and Shparlinski [1] obtained a lower bound for the size of the product of two sets of rational numbers

$$A, B \subset \mathcal{F}_Q = \{q/q' : 1 \leq q, q' \leq Q\}$$

and they applied it to the study of the distribution of elements of multiplicative groups in residue rings. See [3] and [2] for related results and more applications of this useful inequality.

Theorem A (BKSh). *If $A, B \subset \mathcal{F}_Q$ then*

$$(1) \quad |AB| \geq |A||B| \exp \left(-(9 + o(1)) \log Q / \sqrt{\log \log Q} \right),$$

where $o(1) \rightarrow 0$ when $Q \rightarrow \infty$.

For any real numbers $Q, Q' \geq 1$ let $\mathcal{F}_{Q,Q'}$ denotes the set of rational numbers

$$\mathcal{F}_{Q,Q'} = \{q/q' : 1 \leq q \leq Q, 1 \leq q' \leq Q'\}.$$

We give the following result which extends and slightly improves Theorem A.

Theorem 1. *If $A, B \subset \mathcal{F}_{Q,Q'}$ then*

$$|A/B| \geq |A||B| \exp \left(-(2\sqrt{\log 2} + o(1)) \log(QQ') / \sqrt{\log \log(QQ')} \right),$$

where $o(1) \rightarrow 0$ when $QQ' \rightarrow \infty$.

Taking $Q' = Q$ and the set $1/B = \{b^{-1} : b \in B\}$ instead of B we improve the constant in (1).

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Corollary 1. *If $A, B \in \mathcal{F}_Q$, then*

$$|AB| \geq |A||B| \exp \left(-(4\sqrt{\log 2} + o(1)) \log Q / \sqrt{\log \log Q} \right).$$

2. PROOF OF THEOREM 1

For any pair of sets $A, B \subset \mathcal{F}_{Q, Q'}$ and $\gcd(r, s) = 1$ we define the sets

$$\begin{aligned} \mathcal{M}(A \times B, r/s) &= \{(a/a', b/b') \in A \times B : \gcd(a, b) = r, \gcd(a', b') = s\} \\ A_{r/s} &= \{a/a' \in A, r \mid a, s \mid a'\} \\ B_{r/s} &= \{b/b' \in B, r \mid b, s \mid b'\}. \end{aligned}$$

It is clear that $\mathcal{M}(A \times B, r/s) \subset A_{r/s} \times B_{r/s}$, so we have

$$(2) \quad |\mathcal{M}(A \times B, r/s)| \leq |A_{r/s}| |B_{r/s}|.$$

We claim that each $c/d \in A/B$ (assume that $\gcd(c, d) = 1$) has at most $\tau(c)\tau(d)$ representation as

$$(3) \quad \frac{c}{d} = \frac{a/a'}{b/b'}$$

with $(a/a', b/b') \in \mathcal{M}(A \times B, r/s)$. Indeed we observe that (3) implies $\frac{c}{d} = \frac{a_0 b'_0}{b_0 a'_0}$ where $a_0 = a/r$, $b_0 = b/r$, $a'_0 = a_0/s$, $b'_0 = b_0/s$. Since $\gcd(c, d) = 1$ and $\gcd(a_0 b'_0, a'_0 b_0) = 1$ then $c = a_0 b'_0$ and $d = a'_0 b_0$, which proves the claim.

Note that $c = a_0 b'_0 \leq QQ'$ and $d = a'_0 b_0 \leq QQ'$, thus the claim implies the inequality

$$(4) \quad |\mathcal{M}(A, B, r/s)| \leq T^2 |A/B|,$$

where $T = T(QQ')$ and $T(x)$ is the function

$$T(x) = \max_{m \leq x} \tau(m).$$

Using (2), (4) and the well known inequality

$$\sum_{\substack{1 \leq r, s \\ rs \leq x}} 1 \leq x(1 + \log x)$$

we get

$$\begin{aligned} (5) \quad |A||B| &= \sum_{\substack{rs \leq x \\ (r, s) = 1}} |\mathcal{M}(A, B, r/s)| + \sum_{\substack{rs > x \\ (r, s) = 1}} |\mathcal{M}(A, B, r/s)| \\ &\leq T^2 |A/B| x(1 + \log x) + \sum_{\substack{rs > x \\ (r, s) = 1}} |A_{r/s}| |B_{r/s}| \end{aligned}$$

for any real number $x \geq 1$. If x is such that the last sum is less than $|A||B|/2$ then we get

$$(6) \quad |A/B| \geq \frac{|A||B|}{2T^2x(1 + \log x)}.$$

Now we are ready to prove the key Lemma.

Lemma 2. *For any $n \geq 1$ and for any $A, B \in \mathcal{F}_{Q, Q'}$ with real numbers $Q, Q' \geq 1$, we have*

$$(7) \quad |A/B| \geq \frac{|A||B|}{(4T)^{n+1}(QQ')^{1/n}(1 + \log(QQ'))}$$

where $T = \max_{m \leq QQ'} \tau(m)$.

Proof. We proceed by induction on n : trivially, since $|B| \leq QQ'$ we have

$$|A/B| \geq |A| \geq \frac{|A||B|}{QQ'},$$

which proves (7) for $n = 1$. Suppose that Lemma 2 is true for some $n \geq 1$.

If there is r/s such that

$$(8) \quad |A_{r/s}||B_{r/s}| \geq \frac{(QQ')^{\frac{1}{n(n+1)}}}{4T(rs)^{1/n}} |A||B|$$

we use induction for the sets $A_{r/s}, B_{r/s} \in \mathcal{F}_{Q/r, Q'/s}$. By observing that the function $T(x) = \max_{m \leq x} \tau(m)$ is a non decreasing function we have

$$\begin{aligned} |A/B| &\geq |A_{r/s}/B_{r/s}| \\ (\text{by induction hypothesis}) &\geq \frac{|A_{r/s}||B_{r/s}|}{(4T)^{n+1}((Q/r)(Q'/s))^{1/n}(1 + \log((Q/r)(Q'/s)))} \\ (\text{by (8)}) &\geq \frac{|A||B|}{(4T)^{n+2}(QQ')^{1/(n+1)}(1 + \log(QQ'))}. \end{aligned}$$

Thus, we assume that

$$|A_{r/s}||B_{r/s}| < \frac{(QQ')^{\frac{1}{n(n+1)}}}{4T(rs)^{1/n}} |A||B|$$

for any r/s , $(r, s) = 1$. In this case we have

$$\begin{aligned} \sum_{rs > x} |A_{r/s}||B_{r/s}| &\leq \max_{rs > x} (|A_{r/s}||B_{r/s}|)^{1/2} \sum_{rs > x} |A_{r/s}|^{1/2} |B_{r/s}|^{1/2} \\ (9) \quad &\leq \frac{(QQ')^{\frac{1}{2n(n+1)}}}{2T^{1/2}x^{\frac{1}{2n}}} (|A||B|)^{1/2} \left(\sum_{r,s} |A_{r/s}| \right)^{1/2} \left(\sum_{r,s} |B_{r/s}| \right)^{1/2}. \end{aligned}$$

To estimate the sums in the brackets we have

$$(10) \quad \sum_{r,s} |A_{r/s}| = \sum_{q/q' \in A} \sum_{\substack{r,s \\ r|q, s|q'}} 1 \leq \sum_{q/q' \in A} \tau(qq') \leq |A|T.$$

Putting in (9) the estimate (10) and the analogous for $\sum_{r,s} |B_{r/s}|$ we have

$$\sum_{rs > x} |A_{r/s}| |B_{r/s}| \leq |A| |B| \frac{T^{1/2} (QQ')^{\frac{1}{2n(n+1)}}}{2x^{\frac{1}{2n}}}.$$

Taking $x = T^n (QQ')^{\frac{1}{n+1}}$ we get

$$\sum_{rs > x} |A_{r/s}| |B_{r/s}| \leq |A| |B| / 2.$$

Then (6) applies and noting that $\log x \leq \log((QQ')^{n+\frac{1}{n+1}}) \leq 2n \log(QQ')$ we get

$$\begin{aligned} |A/B| &\geq \frac{|A| |B|}{2T^2 x (1 + \log x)} \\ &\geq \frac{|A| |B|}{2T^{n+2} (QQ')^{\frac{1}{(n+1)}} (1 + 2n \log(QQ'))} \\ &\geq \frac{|A| |B|}{(4T)^{n+2} (QQ')^{\frac{1}{(n+1)}} (1 + \log(QQ'))} \times \frac{2^{2n+3} (1 + \log(QQ'))}{1 + 2n \log(QQ')} \\ &\geq \frac{|A| |B|}{(4T)^{n+2} (QQ')^{\frac{1}{(n+1)}} (1 + \log(QQ'))}. \end{aligned}$$

□

The well known upper bound for the divisor function,

$$\tau(m) \leq \exp((\log 2 + o(1)) \log m / \log \log m)$$

implies

$$T \leq \exp((\log 2 + o(1)) \log(QQ') / \log \log(QQ')).$$

Thus, an optimal choice of n in Lemma 2 is $n \sim \sqrt{\frac{\log \log(QQ')}{\log 2}}$, from where Theorem 1 follows.

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